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Research Article

Extended Coprime Graph of Finite Group

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Abstract

Algebraic graph theory encompasses an interface between group and graph which proves to be one of the most productive areas. The concept of coprime graph was introduced by M. Sattanathan and R. Kala (2009). In this paper, we introduce a new graph namely extended coprime graph from the finite group and discuss some properties of extended coprime graph.

Keywords: Graph, Group, Coprime Graph of a Group, Extended Coprime Graph

1. Introduction

Algebraic graph theory plays a leading role in modern Mathematics. A lot of studies are being done by connecting both groups and graphs, which have applications in the area like Computer Science, Chemistry and Engineering. In literature we can find various ways of associating groups and graph. Some of the examples are cayley graphs [1], prime graphs [2], power graphs [3]. The notion of coprime graph of a group was introduced by Ma et al. [4], in which elements of a group act as the vertices of the graph and any two vertices are adjacent if the order of the corresponding vertices are relatively prime. Numerous studies are still going on by this way.

In [5], R. Rajendra and P. Siva Kota Reddy define the concept of General Order Prime Graph of a group. From this motivation we define a new graph namely extended coprime graph of group G having finite order and it is denoted by ${}^{E}\Gamma_{G}$.

Our aim is to introduce the concept of extended coprime graph and try to discuss some of its properties. In section 2 we come across with some of the basic concepts. Section 3 we introduce a new graph namely extended coprime graph with sufficient examples. Then in section 4 we talk about some properties of extended coprime graph. And in section 5 we discuss the theorems on the isomorphism of extended coprime graphs. All groups (G) and graphs (Γ) considered throughout this paper are finite and we denote prime numbers as p.

2. Preliminaries

Here in this section, we collect some definitions and results which are useful in the upcoming sections.

Theorem 2.1: [6 , 7] The order of an element of a finite group divides the order of the group.

Definition 2.2: [7] A mapping ϕ from a group G into a group G' is said to be a homomorphism if for all a, b $\in$ G, ϕ(ab) = ϕ(a) ϕ(b). A homomorphism ϕ from G into G' is said to be an isomorphism if ϕ is one to one. By an automorphism of a group G we shall mean an isomorphism of G onto itself.

Definition 2.3: [8] A graph H is called a subgraph of G if V(H) $\subseteq$ V(G), E(H) $\subseteq$ E(G).A subgraph H of G is a spanning subgraph of G, if V(H) = V(G).

Definition 2.4: [9] For a pair of vertices u and v of a graph Γ, the length of any shortest path between u and v of a connected graph Γ is called the distance between u and v and is denoted by d(u, v). The diameter of Γ is defined as max {d (u ,v)| u ,v $\in$ V(Γ)} and is denoted by diam(Γ). The girth of a graph Γ is the length of a smallest cycle in Γ.

Definition 2.5: [9] Two graph G = (V,E) and H =(U, F) are isomorphic iff there is a bijection $\phi: V \rightarrow U$ such that for any pair u,v in V we have uv $\in$ E iff $\phi(u)\phi(v) \in F$. Then ϕ is called an isomorphism of G onto H and ϕ^{-1} an isomorphism of H onto G. We use the notation G $\cong$ H to denote that G is isomorphic to H. An isomorphism of G onto itself is called automorphism.

Definition 2.6: [10] The coprime graph of a group G, denoted by Γ_G ,is graph whose vertices are elements of G and two distinct vertices x and y are adjacent if and only if (|x|, |y|) =1.

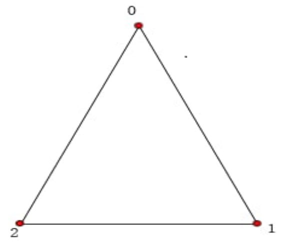
3. Extended Coprime Graph of Finite Group

Here we try to introduce the concept of extended coprime graph and explain it using sufficient examples.

Definitions 3.1: The extended coprime graph of a group G of order n (denoted by ${}^{E}\Gamma_G$) is a graph whose set of vertices is G itself and any two different vertices r_1 and r_2 are adjacent if and only if ($|r_1|$, $|r_2|$) = 1 or p where p is a prime and p $\leq$ n.

Note: We do not consider self- loops in ${}^{E}\Gamma_G$ through some time we have ($|r_1|$,$|r_1|$) = 1 or a prime p, p $\leq$ n

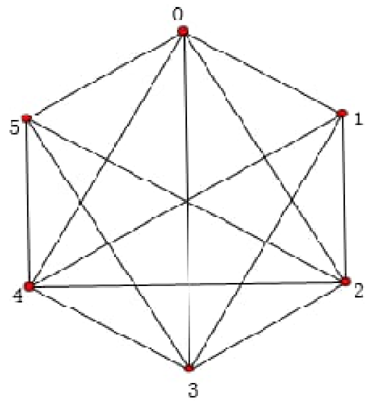
Example 3.2. Consider the group G = Z_3. The graph ${}^{E}\Gamma_G$ has the set of vertices V ={0, 1, 2 }.
Here | 0 | = 1, | 1 | = | 2 | = 3



Example 3.3: Consider the group G = Z_5.

The graph ${}^{E}\Gamma_G$ has the set of vertices V = { 0, 1, 2, 3, 4, 5 }

Here |0|= 1, |1|= 6, |2|=3, |3|=2, |4|=3 |5|=6



Observation : For a group G , the **e**xtended coprime graph of is always a simple.

Theorem 3.4: For a group G, coprime graph Γ_G is a spanned subgraph of extended coprime graph ${}^{E}\Gamma_G$.

Proof: The theorem can be proved exactly from definition 2.6 of coprime graph and definition 3.1 of extended coprime graph.

Remark: Coprime graph equals extended coprime graph only when the group is of order 2.

Theorem 3.5: For any group G of order n, extended coprime graph ${}^{E}\Gamma_G$ is connected and its maximum degree $\Delta({}^{E}\Gamma_G) = n - 1$.

Proof: Given, group G of order n.

The only element of G with order 1 is the identity element .

Vertex associated with identity element in ${}^{E}\Gamma_G$ is a neighbour of all other (n-1) vertices (a vertex r_1 is a neighbor of r_2 in a graph, if r_1r_2 is an edge in that graph).

Therefore ${}^{E}\Gamma_G$ is always a connected graph.

It is obvious from the above explanation that $\Delta({}^{E}\Gamma_G) = (n-1)$.

4. Properties of Extended Coprime Graph

Here we investigate some properties of extended coprime graph.

Theorem 4.1 : The extended coprime graph ${}^{E}\Gamma_G$ for a group G of prime order is always a complete graph.
Proof: Given | G | =p , where p prime.
Then V(${}^{E}\Gamma_G$) = { e, r_1, r_2,.. r_{p-1}}
Clearly | e | = 1 and | r_i | = p for every i.
Then (| e | ,| r_i |) = 1 and (| r_i |,| r_j |) = p for i ≠ j.
Thus every pair of different vertices of ${}^{E}\Gamma_G$ are adjacent. So the graph is a complete graph.

Corollary 4.2: ${}^{E}\Gamma_G$ is regular only when group G is of prime order .
Proof: By using Theorem 4.1, we get deg (a) = (p – 1) $\forall$ a $\in$ V(${}^{E}\Gamma_G$)
Therefore ${}^{E}\Gamma_G$ is a regular graph.

Theorem 4.3: Let G be any group, then diameter of ${}^{E}\Gamma_G$ is always less than or equal to 2.
Proof: Let ${}^{E}\Gamma_G$ be its extended coprime graph of the group G having order n.
Consider 2 distinct vertices r_1 , r_2 of ${}^{E}\Gamma_G$.
If (| r_1 |,| r_2 |) = 1 or p where p prime p $\leq$ n. Then r_1 and r_2 are adjacent and hence d(r_1,r_2)=1.
Consequently we assume that r_1 and r_2 are non adjacent.
Then (| r_1 |,| r_2 |) $\neq$ 1 or p where p prime p $\leq$ n. Clearly the identity element e is a neighbour of all other vertices of ${}^{E}\Gamma_G$.
As (| e |,| r_1 |)=1 and (| e |,| r_2 |)=1, we obtain vertex e is adjacent to both r_1 and r_2 and hence d(r_1,r_2)=2.
Thus we get, diam (${}^{E}\Gamma_G$) $\leq$ 2.

Theorem 4.4 : For a group G , diam (${}^{E}\Gamma_G$) = 1 if and only if G is isomorphic to $\mathbf{Z}_p$.
Proof: Suppose |G| = n, where n is not a prime number.
Assume diam (${}^{E}\Gamma_G$) = 1. Then there exist at least two distinct non-identity element r_1 and r_2 of G such that r_1 and r_2 are adjacent in the extended coprime graph ${}^{E}\Gamma_G$.
i.e,(| r_1 |,| r_2 |) = 1 or p where p prime p $\leq$ n.
This follows that either |r_1| > 2 or |r_2| > 2.
If |r_1| = n_1, where n_1 is not a prime number and 2 < n_1 $\leq$ n. Clearly ${}^{E}\Gamma_G$ has ϕ(n_1) > 2 number of elements of order n_1.
Let s_1 and s_2 be elements of G of order n_1. Then (| s_1 |,| s_2 |) = n_1.
Thereby s_1 is not adjacent to s_2 in ${}^{E}\Gamma_Q$. Therefore d(s_1 ,s_2) $\neq$ 1, which is a contradiction.
Thus G is isomorphic to $\mathbf{Z}_p$.
The converse is clear.

Theorem 4.5: The girth of extended coprime graph equals three, for any finite group having order greater than 2.
Proof: Let $r_1 \neq$ e, r_2 $\neq$ e $\in$ G such that (| r_1 |,| r_2 |) = 1 or p where p prime p $\leq$ n. Thus r_1 and r_2 are joined by an edge.
Also by Theorem 3.5, identity element e is a neighbour of all other vertices of ${}^{E}\Gamma_G$.

Hence $\{r_1, r_2, e\}$ forms a 3- cycle in ${}^{E}\Gamma_G$.And so the girth of ${}^{E}\Gamma_G$ equals 3.

Theorem 4.6: For any group G with order greater than 3, ${}^{E}\Gamma_G$ can never be a unicyclic graph.

Proof: Consider a group G with $|G| > 3$. Suppose ${}^{E}\Gamma_G$ is unicyclic.

By using Theorem 3.5, $\Delta({}^{E}\Gamma_G) = n-1$. Then ${}^{E}\Gamma_G - \{e\}$ has exactly one edge e_1.

Let $e_1 = r_1r_2$, then $(|r_1|,|r_2|) = 1$ or p where p is a prime and $p \le n$.

In particular $(|r_1|,|r_2|) = p_1$.

Then at least one of r_1 and r_2 should not be a self- inverse element.

Let $(r_1)^{-1} \neq r_1$ be the inverse of r_1. Clearly $|r_1| = |(r_1)^{-1}|$ and hence $(r_1)^{-1}$, r_2 are also adjacent in ${}^{E}\Gamma_G - \{e\}$.

Remark: Extended coprime graph ${}^{E}\Gamma_G$ is unicyclic only when $G \cong Z_3$

Theorem 4.7 : The extended coprime graph ${}^{E}\Gamma_{D2n}$ is a complete for a prime n.

Proof: Suppose n is prime.

The dihedral group $D_{2n} = \{e, r, r^2, \ldots, r^{n-1}, s, rs, r^2s, \ldots, r^{n-1}s\}$ ie { n rotations , n reflections }

Therefore $|e| = 1$, $|r| = |r^2| = \ldots = |r^{n-1}| = n$

$|s| = |rs| = |r^2s| = \ldots = |r^{n-1}s| = 2$

From this it is clear that for any two vertices r_1 , r_2 of ${}^{E}\Gamma_{D2n}$

$(|r_1|,|r_2|) = 1$ or 2 or n

Hence any two vertices of ${}^{E}\Gamma_{D2n}$ is adjacent.

Theorem 4.8: For a group G with $|G| = m = p_1^{n1}p_2^{n2}\ldots p_r^{nr}$. Let q be the number of edges of the graph ${}^{E}\Gamma_G$. Then $(r+1)\{m - \frac{r+2}{2}\} \le q \le \frac{m(m-1)}{2}$.

Proof: From the definition of extended coprime graph, ${}^{E}\Gamma_G$ is a simple graph.

Given that q be the number of edges of the graph ${}^{E}\Gamma_G$.

Maximum number of edges is obtained if ${}^{E}\Gamma_G$ is complete graph. This happens when order of G is a prime number. Therefore $q \le \frac{m(m-1)}{2}$.

Also given $|G| = m = p_1^{n1}p_2^{n2}\ldots p_r^{nr}$.

Clearly identity element is a neighbour of other (m-1) vertices of ${}^{E}\Gamma_G$.

An element having order p_1 is a neighbour of other (m-2) vertices of ${}^{E}\Gamma_G$.

Continuing like this an element having order p_r is a neighbour of other (m-(r+1)) vertices.

Hence minimum number of edges of ${}^{E}\Gamma_G$ is

$$(m-1) + (m-2) + \ldots + (m-(r+1))$$
$$= (r+1)m - [1 + 2 + \ldots. + (r+1)]$$
$$= (r+1)m - \frac{(r+1)(r+2)}{2}$$
$$= (r+1)\{m - \frac{r+2}{2}\}$$

Therefore (r+1) $\{m - \frac{r+2}{2}\} \leq q$

5. Isomorphism of Extended Coprime Graph

Here we study about the isomorphism and automorphism of extended coprime graphs.

Theorem 5.1 : For any two groups G_1 and G_2 , if $G_1 \cong G_2$, then ${}^E\Gamma_{G_1} \cong {}^E\Gamma_{G_2}$.

Proof : Let ϕ be an isomorphism from G_1 to G_2.

Let $|G_1| = |G_2| = n$.

Now two vertices r_1 and r_2 of ${}^E\Gamma_{G_1}$ are adjacent.

$\Leftrightarrow$ ($|r_1|, |r_2|$) = 1 or p where $p \leq n$.

$\Leftrightarrow$ ($|\phi(r_1)|, |\phi(r_2)|$) = 1 or p where $p \leq n$.

$\Leftrightarrow$ $\phi(r_1)$ and $\phi(r_2)$ are adjacent in ${}^E\Gamma_{G_2}$.

Thus ${}^E\Gamma_{G_1} \cong {}^E\Gamma_{G_2}$.

Note : Identity function I is an automorphism from ${}^E\Gamma_G$ to itself.

Theorem 5.2: Let G be a finite group, then Aut(G) $\subseteq$ Aut(${}^E\Gamma_G$).

Proof: Let G be a group

$\phi \in$ Aut(G) $\Rightarrow \phi$ be an isomorphism from G onto G

Now two vertices r_1 and r_2 of ${}^E\Gamma_G$ are adjacent.

$\Rightarrow$ ($|r_1|, |r_2|$) = 1 or p where $p \leq n$.

$\Rightarrow$ ($|\phi(r_1)|, |\phi(r_2)|$) = 1 or p where $p \leq n$.

$\Rightarrow \phi(r_1)$ and $\phi(r_2)$ are adjacent in ${}^E\Gamma_G$.

$\Rightarrow \phi$ be an isomorphism from ${}^E\Gamma_G$ onto ${}^E\Gamma_G$.

$\Rightarrow \phi \in$ Aut(${}^E\Gamma_G$).

Hence Aut(G) $\subseteq$ Aut(${}^E\Gamma_G$).

Theorem 5.3: The collection of all automorphisms of extended coprime graph form a group under the binary operation composition.

Proof: We denote Aut(${}^E\Gamma_G$) as the collection of all automorphisms of extended coprime graph. To prove (Aut(${}^E\Gamma_G$), $\circ$) is a group, where $\circ$ denote the binary operation composition.

1. If ϕ_1 , ϕ_2 $\in$ Aut(${}^E\Gamma_G$)

   Then ϕ_1 and ϕ_2 be bijections on V(${}^E\Gamma_G$) which preserves adjacency and non adjacency.

   Clearly $\phi_1 \circ \phi_2$ is also a bijection on V(${}^E\Gamma_G$).

   Thus if r_1 and r_2 are adjacent in ${}^E\Gamma_G$.

   $\Rightarrow \phi_2(r_1)$ and $\phi_2(r_2)$ are adjacent in ${}^E\Gamma_G$.

   $\Rightarrow \phi_1(\phi_2(r_1))$ and $\phi_1(\phi_2(r_2))$ are adjacent in ${}^E\Gamma_G$.

   $\Rightarrow \phi_1 \circ \phi_2(r_1)$ and $\phi_1 \circ \phi_2(r_2)$ are adjacent in ${}^E\Gamma_G$.

Hence, we show that $\phi_1 \circ \phi_2$ preserves adjacency. Similarly we can prove that it preserves non adjacency. Therefore $\phi_1 \circ \phi_2 \in$ Aut(${}^{E}\Gamma_G$).

2. Binary operation composition on Aut(${}^{E}\Gamma_G$) satisfies associativity.
3. For any $\phi \in$ Aut(${}^{E}\Gamma_G$), $\phi \circ I = I \circ \phi = \phi$. Hence I is the identity element of Aut(${}^{E}\Gamma_G$).
4. For any $\phi \in$ Aut(${}^{E}\Gamma_G$), we can find $\phi^{-1} \in$ Aut(${}^{E}\Gamma_G$), such that $\phi^{-1} \circ \phi = \phi \circ \phi^{-1} =$ I.

Thus (Aut(${}^{E}\Gamma_G$), $\circ$) satisfies all the four conditions , so it forms a group.

6. Conclusion

This paper tries to introduce the new concept namely extended coprime graph. And also try to discuss some main theorems and some of its properties connecting isomorphism.

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